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DESIGN OF *M*-CHANNEL PERFECT RECONSTRUCTION FILTERBANKS WITH IIR-FIR HYBRID BUILDING BLOCKS

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ABSTRACT

This paper discusses a new structure of *M*-channel IIR perfect reconstruction filterbanks. A novel design implementation for the new building block defined as a product of an IIR building block and an FIR building block is presented. The IIR building blocks are written by state space representation, in which we can easily obtain a stability of the filterbank by placing its eigen values inside the unit circle. Due to cascading of building blocks, we can get more free parameters. We introduce the condition how we obtain the new building blocks without increasing of the filter order. Additionally, by showing the simulation results, we show that our proposed FBs have better stopband attenuation than a conventional method.

Index Terms—*M*-channel Infinite Impulse Response Perfect Reconstruction Filterbanks, State Space Representation, FIR-IIR hybrid building blocks

1. INTRODUCTION

Recently, many researchers have been studying multirate signal processing. One of the most efficient techniques for processing wideband digital signals in communication systems and compressing audio, image and video signals is called as filter bank (FB). A perfect reconstruction (PR) FB design involves its analysis and synthesis polyphase matrices $\mathbf{E}(z)$ and $\mathbf{R}(z)$ [1]. These matrices consist of polynomials of z which is called delay. There are two types of transfer function (TF) of FBs, called finite impulse response (FIR) and infinite impulse response (IIR). In an FIR case, the denominator of TF is 1, so we do not need to consider its stability. On the contrary, in an IIR case, the denominator involves delays. Thus, it is necessary to consider its stability. Meanwhile, IIR FBs have superiority in terms of the order of the TF. In other word, they can be realized in lower order (much fewer multipliers and adders) than FIR FBs to obtain the desired response specifications [2]. While there are many well-developed FIR FB design theories, there are few satisfactory implementations for IIR FB design because of the problem of its stability. Several design methods for the case of IIR PRFBs whose analysis FB is causal stable (poles of $\mathbf{E}(z)$ are inside the unit circle) and the synthesis FB is anti-causal stable (poles of $\mathbf{R}(z)$ are outside the unit circle) have already been introduced. For example, [3] proposed using all pass filters instead of delays in lossless matrices to design IIR PRFB, approximately linear-phase filters using complex all pass sections is proposed in [4], and orthogonal IIR 2-channel PRFBs are designed using all pass filters in [5]. For such FBs, synthesis filtering needs to be performed in anti-causal fashion by introducing time reversal [6], which increases the storage cost of the system and may only be acceptable for image processing applications.

Although some design methods have been proposed for the causal and stable case [7], they mostly treated the 2-channel FB cases. Recently, an *M*-channel real IIR FB [8] was presented, but the design method includes a complicated stabilization procedure of the synthesis filters.

In this paper, we introduce a class of IIR causal stable PRFBs obtained by cascading of FIR and IIR building blocks. Firstly, we present the conventional method for IIR PRFB by using state space representation [9] and the structure of FIR degree-1 building blocks in section 2. In section 3, we show the condition how we obtain the new building block without increasing the order for the case of order-1 and 2. Frequency responses of the proposed FBs are presented and compared with the conventional method in section 4.

Notations: Boldface small and capital letters represent vectors and matrices, respectively. $det()$, $adj()$ and $trace()$ denote determinant, adjoint and trace of a matrix.

2. REVIEW

2.1 Polyphase Structure for Filterbanks

Fig. 1 shows a typical structure of an *M*-channel maximally decimated FB, where $H_k(z)$ and $F_k(z)$ are the k-th (for $k = 0 \cdots M-1$) analysis and synthesis filter, respectively. In Fig. 2, the analysis and synthesis filters are shown by using the polyphase matrices $\mathbf{E}(z)$ and $\mathbf{R}(z)$ represented as follows:

$$\begin{aligned} [H_0(z)\ H_1(z)\ \cdots\ H_{M-1}(z)]^T &= \mathbf{E}(z^M)\mathbf{e}(z)^T \\ [F_0(z)\ F_1(z)\ \cdots\ F_{M-1}(z)] &= \mathbf{e}(z)\mathbf{R}(z^M) \\ \mathbf{e}(z) &= [1\ z^{-1}\ \cdots\ z^{-(M-1)}]. \end{aligned} \quad (1)$$

It is clear that the condition for PR is $\mathbf{R}(z)\mathbf{E}(z) = c\mathbf{I} \quad (c \in \mathbb{R})$ (PR condition [1]).

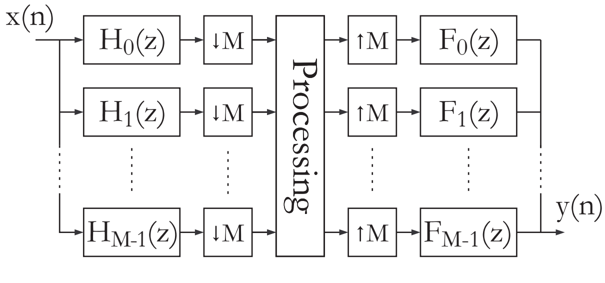

Figure 1: An *M*-channel filterbank.

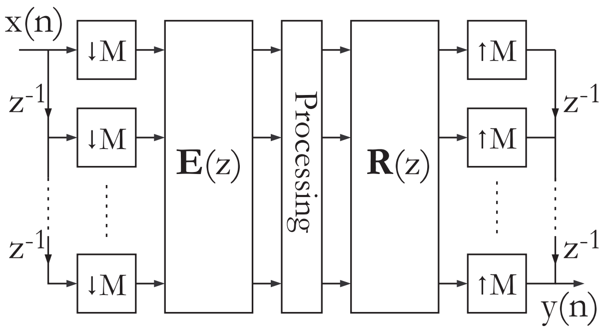

Figure 2: The polyphase representation of FBs.

2.2 IIR Filterbanks Using State Space Structure

The PR condition for general $M \times M$ polyphase matrices $\mathbf{E}(z)$ and $\mathbf{R}(z)$ are

$$\mathbf{R}(z) = \mathbf{E}^{-1}(z) = \frac{adj(\mathbf{E}(z))}{det(\mathbf{E}(z))} \tag{2}$$

where obviously $\mathbf{E}(z)$ is nonsingular. For IIR FBs, causal stable synthesis filters are obtained if $\det(\mathbf{E}(z))$ is minimum phase with $\mathbf{E}(z)$ being causal. The problem of constraining $\mathbf{E}(z)$ with minimum phase determinant is resolved by considering the minimal factorization of $\mathbf{E}(z)$ in state space structure.

As it is known, any rational function $\mathbf{E}(z)$ can be expressed in state space structure [9],

$$\mathbf{E}(z) = \mathbf{D} + \mathbf{C}'(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}'$$

where $\mathbf{D}$, $\mathbf{A}$, $\mathbf{B}'$and $\mathbf{C}'$ are $M \times M$, $m \times m$, $m \times M$ and $M \times m$ matrices respectively $(m \leq M)$. $\mathbf{A}$ is called the state transition matrix. Then to simplify, we can rewrite the above equation

$$\mathbf{E}(z) = \mathbf{D}(\mathbf{I} + \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}) = \mathbf{D}\mathbf{E}'(z) \tag{3}$$

And the synthesis polyphase matrix **R**(z) is given by

$$\mathbf{R}(z) = (\mathbf{I} - \mathbf{C}(z\mathbf{I} - \mathbf{A}^*)^{-1}\mathbf{B})\mathbf{D}^{-1} = \mathbf{R}'(z)\mathbf{D}^{-1} \tag{4}$$

where

$$\mathbf{A}^* = \mathbf{A} - \mathbf{B}\mathbf{C}. \tag{5}$$

Since $\mathbf{R}(z)\mathbf{E}(z) = \mathbf{I}$, this implies $\mathbf{R}'(z)\mathbf{E}'(z) = \mathbf{I}$. It is sufficient to focus on factorization of $\mathbf{E}(z)$ and $\mathbf{R}(z)$. The structure of the building block is shown in Fig. 3 (a) and (b). Because $\mathbf{R}(z)$ can be calculated by matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ of $\mathbf{E}(z)$ uniquely, in this paper, we focus on only the structure of the building block in $\mathbf{E}(z)$

- *Free parameters*
  There is no restriction of every matrix $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$. These are all nonsingular matrices. Thus, the number of parameters is $M^2 + 2mM + m^2$.

2.3 Controllability and Observability

In designing FBs using state space representation, we have to impose the both conditions observability and controllability to obtain minimal system $\mathbf{E}(z)$[10].

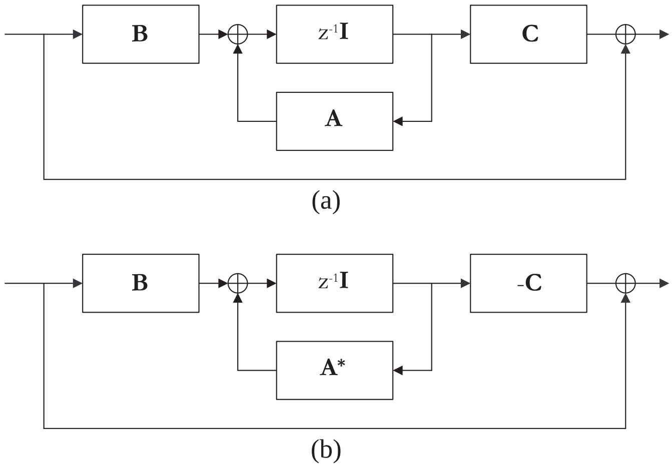

Figure 3: A realization of the conventional filterbanks (a)$\mathbf{E}'(z)$ (b)$\mathbf{R}'(z)$

- *Controllability condition*
  The $m \times Mk$ matrix

  $$\mathscr{C}(\mathbf{A},\mathbf{B}) = [\mathbf{B}\ \mathbf{A}\mathbf{B}\ \mathbf{A}^2\mathbf{B}\ \cdots\ \mathbf{A}^k\mathbf{B}]$$

  must be of rank m; this condition is called *controllability condition.*

- *Observability condition*
  The $m \times Mk$ matrix

  $$\mathscr{O}(\mathbf{C},\mathbf{A}) = [\mathbf{C}^T\ (\mathbf{C}\mathbf{A})^T\ (\mathbf{C}\mathbf{A}^2)^T \cdots\ (\mathbf{C}\mathbf{A}^k)^T]$$

  must be of rank m; this condition is called *observability condition.*

To keep $\mathbf{E}(z)$ minimal, additional constraints have to be imposed in the present design. It can be seen that if $m \leq M$, full rank matrices $\mathbf{B}$ and $\mathbf{C}$ are enough to satisfy both minimality conditions discussed above, irrespective of the rank of $\mathbf{A}$. So, in the present design methods we assume $m \leq M$ (the dimension of the matrix $\mathbf{A}$ never exceeds the number of channels M) and $rank(\mathbf{B}) = rank(\mathbf{C}) = m$.

2.4 FIR Degree-1 Building Block

A class of causal M-channel FIR biorthogonal (BO) FBs of order-L are factorized into [11]

$$\mathbf{E}(z) = \mathbf{W}_L(z) \cdots \mathbf{W}_1(z)\mathbf{E}_0 \quad (j = 1,2,\cdots m) \tag{6}$$

where $\mathbf{E}_0$ is an $M \times M$ nonsingular matrix and which have an FIR, anticausal inverse. In the above equation, each $\mathbf{W}_m(z)$ is a first-order BO building block given by

$$\mathbf{W}_m(z) = \mathbf{I} - \mathbf{U}_m\mathbf{V}_m^{\dagger} + z^{-1}\mathbf{U}_m\mathbf{V}_m^{\dagger} \tag{7}$$

where $\cdot^{\dagger}$ denotes conjugate transpose and the $M \times \gamma_m$ parameter matrices $\mathbf{U}_m$ and $\mathbf{V}_m$ satisfy

$$\mathbf{V}_m^{\dagger}\mathbf{U}_m = \begin{bmatrix} 1 & \times & \cdots & \times \\ 0 & 1 & \cdots & \times \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{\gamma_m \times \gamma_m}$$

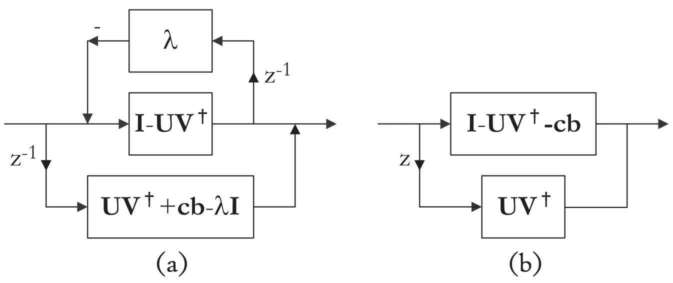

Figure 4: The structure of Order-1 IIR-FIR hybrid building blocks:(a) analysis filter (b) synthesis filter.

for some integer $1 \le \gamma_m \le M$, where $\times$ indicates possibly nonzero elements. This is a generalization of the paraunitary order-one factorization given in [12] where $\mathbf{U}_m = \mathbf{V}_m$, and has been used for factoring the biorthogonal lapped transform (BOLT) [13].

- Since the rank of $\mathbf{V}_m^\dagger \mathbf{U}_m$ is γ_m, the McMillan degree of $\mathbf{W}_m(z)$ as in (7) is γ_m.
- The structure in (6) completely spans all causal FIR PRFBs having anticausal FIR inverses. The spanned analysis filters have filter lengths no greater than $M(L+1)$, and the McMillan degree of $\mathbf{E}(z)$ ranges from L to ML, where L is the order of the FB.
- The Type-II synthesis polyphase matrix $\mathbf{R}(z)$ is given by

$$\mathbf{R}(z) = \mathbf{E}_0^{-1}\mathbf{W}_1^{-1}(z)\cdots\mathbf{W}_L^{-1}(z) \tag{8}$$

where $\mathbf{W}_m^{-1}(z^{-1}) = \mathbf{I} - \mathbf{U}_m\mathbf{V}_m + z\mathbf{U}_m\mathbf{V}_m$, which is anticausal and satisfies $\mathbf{R}(z)\mathbf{E}(z) = \mathbf{I}$ for PR. Due to the possibly nonzero off-diagonal elements of $\mathbf{V}_m^\dagger\mathbf{U}_m$, the order of $\mathbf{W}_m^{-1}(z^{-1})$ can be greater than one, and thus the synthesis bank can have filter lengths different from $M(L+1)$.

3. PRFBS WITH IIR-FIR HYBRID BUILDING BLOCKS

In this section, we introduce a novel building block obtained by product of the conventional IIR building block and the FIR BO building block. This paper considers the cases where the number of degrees of the new building block is one or two.

3.1 Design of Order-1 PRFBs

We consider the condition for the case of order-1 ($m = 1$) in this part. Analysis matrix $\mathbf{E}'(z)$ can be rewritten by

$$\begin{aligned} \mathbf{E}'(z) &= \mathbf{I} + \mathbf{c}(z-\lambda)^{-1}\mathbf{b} \\ &= \frac{\mathbf{I} - z^{-1}(\mathbf{cb} - \lambda\mathbf{I})}{1-\lambda z^{-1}} \end{aligned} \tag{9}$$

where $\mathbf{b}$ and $\mathbf{c}$ are a row and a column vector respectively. Since λ is pole, we obtain the stable filterbank if λ is placed inside unit circle. Our proposed building block is represented as product of (7) and (9) as

$$\begin{aligned} \mathbf{G}(z) &= \frac{\mathbf{I} - z^{-1}(\mathbf{cb} - \lambda\mathbf{I})}{1-\lambda z^{-1}} \cdot (\mathbf{I} - \mathbf{UV}^\dagger + z^{-1}\mathbf{UV}^\dagger) \\ &= \frac{\mathbf{I} - \mathbf{UV}^\dagger + z^{-1}\mathbf{G}_1 - z^{-2}\mathbf{G}_2}{1-\lambda z^{-1}} \end{aligned}$$

where the size of $\mathbf{U}$ and $\mathbf{V}$ is $M \times 1$ and

$$\begin{aligned} \mathbf{G}_1 &= \mathbf{UV}^\dagger - \mathbf{cb} + \lambda\mathbf{I} + (\mathbf{cb} - \lambda\mathbf{I})\mathbf{UV}^\dagger, \\ \mathbf{G}_2 &= (\mathbf{cb} - \lambda\mathbf{I})\mathbf{UV}^\dagger. \end{aligned}$$

So as not to increase the order of the building block, the condition is

$$(\mathbf{cb} - \lambda\mathbf{I})\mathbf{U} = 0.$$

Then it is rewritten by

$$\begin{cases} \lambda = \mathbf{bc} \\ \mathbf{U} = \mathbf{c} \end{cases}. \tag{10}$$

Polyphase matrices $\mathbf{E}(z)$ and $\mathbf{R}(z)$ using proposed building blocks are expressed as

$$\mathbf{E}(z) = \mathbf{D}\frac{\mathbf{I} - \mathbf{UV}^\dagger + z^{-1}(\mathbf{UV}^\dagger + \mathbf{cb}^\dagger - \lambda\mathbf{I})}{1-\lambda z^{-1}} \tag{11}$$

$$\mathbf{R}(z) = [\mathbf{I} - \mathbf{UV}^\dagger - \mathbf{cb} + z\mathbf{UV}^\dagger]\mathbf{D}^{-1}. \tag{12}$$

As shown in (11) and (12), the systems of FB are IIR system for $\mathbf{E}(z)$ and FIR system for $\mathbf{R}(z)$. Fig. 4 (a) and (b) show the structures of Order-1 IIR-FIR Hybrid building blocks.

- *Free parameters*
  In the conventional method, there are $(M+1)^2$ parameters for the case of Order-1. In our method, a restriction imposed for λ which is same as matrix $\mathbf{A}$ in (3). There are no restrictions of matrices $\mathbf{D}$, $\mathbf{b}$ and $\mathbf{c}$. Thus, the total number of free parameters is $M^2 + 3m - 1$.

3.2 Design of Order-2 PRFBs

In this subsection, we introduce the new design for the case of order-2 ($m = 2$) similar to order-1. We denote the proposed building block of analysis polyphase matrix as follows:

$$\mathbf{G}(z) = [\mathbf{I} + \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}][\mathbf{I} - \mathbf{UV}^\dagger + z^{-1}\mathbf{UV}^\dagger]$$

where the size of $\mathbf{U}$ and $\mathbf{V}$ is $M \times 2$. Then $\mathbf{G}(z)$ can be rewritten as

$$\mathbf{G}(z) = \frac{\mathbf{I} + z^{-1}\mathbf{G}_1 + z^{-2}\mathbf{G}_2}{1 - trace(\mathbf{A})z^{-1} + det(\mathbf{A})z^{-2}}[\mathbf{I} - \mathbf{UV}^\dagger + z^{-1}\mathbf{UV}^\dagger]$$

where

$$\begin{aligned} \mathbf{G}_1 &= \mathbf{CB} - trace(\mathbf{A})\mathbf{I}, \\ \mathbf{G}_2 &= det(\mathbf{A})(\mathbf{I} - \mathbf{CA}^{-1}\mathbf{B}). \end{aligned}$$

The constraints can be expressed as

$$det(\mathbf{A})(\mathbf{I} - \mathbf{CA}^{-1}\mathbf{B})\mathbf{UV}^\dagger = \mathbf{0}.$$

then it is rewritten by

$$\begin{cases} \mathbf{A} = \mathbf{BC} \\ \mathbf{U} = \mathbf{C} \end{cases}. \tag{13}$$

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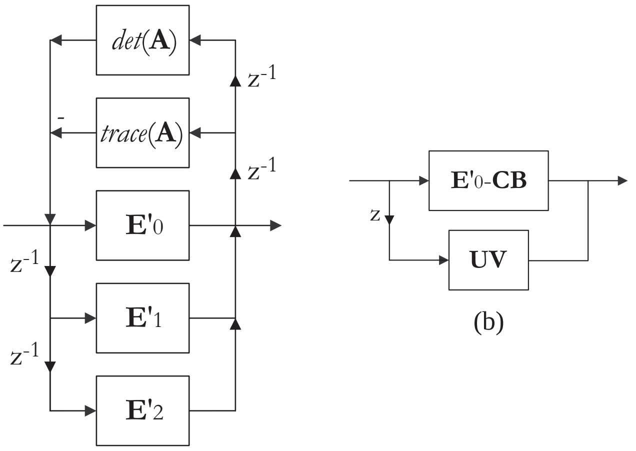


(a) (b)

Figure 5: The structure of IIR-FIR hybrid building blocks: (a) analysis filter (b) synthesis filter.

Hence, our proposed $\mathbf{E}(z)$ and $\mathbf{R}(z)$ are represented as

$$\mathbf{E}(z) = \mathbf{D}\frac{\mathbf{E}'_0 + z^{-1}\mathbf{E}'_1 + z^{-2}\mathbf{E}'_2}{1 - trace(\mathbf{A})z^{-1} + det(\mathbf{A})z^{-2}} \qquad (14)$$

$$\mathbf{R}(z) = [\mathbf{E}'_0 - \mathbf{CB} + z\mathbf{UV}^{\dagger}]\mathbf{D}^{-1} \qquad (15)$$

where

$$\mathbf{E}'_0 = \mathbf{I} - \mathbf{UV}^{\dagger}$$
$$\mathbf{E}'_1 = \{\mathbf{CB} - trace(\mathbf{A})\mathbf{I}\}\mathbf{E}_0 + \mathbf{UV}^{\dagger}$$
$$\mathbf{E}'_2 = det(\mathbf{A})(\mathbf{I} - \mathbf{CA}^{-1}\mathbf{B}) - \{\mathbf{CB} - trace(\mathbf{A})\mathbf{I}\}.$$

As it is mentioned above, filterbanks of the proposed method have both IIR and FIR filters. By cascading these new building blocks with order-1 and 2, we can obtain a higher order filterbank. Fig. 5 shows the structure of analysis and synthesis IIR-FIR hybrid building blocks.

- *Free parameters*
  In the conventional method, the number of free parameters is $M^2 + 4M + 4$ for the case of Order-2. In proposed method, there is some restrictions same as the case of Order-1. Thus, we have $M^2 + 6M - 3$ parameters totall1y.

4. RESULTS

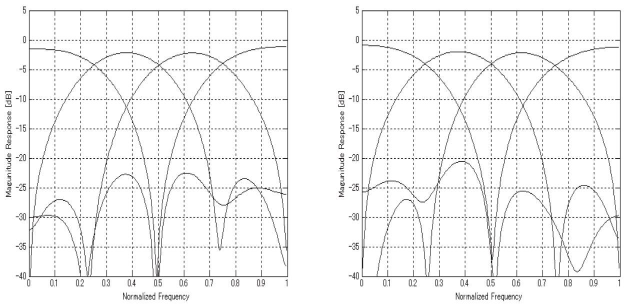
In this section, we present the design examples of proposed IIR PRFBs. In this paper, we optimize the cost function Φ to design FB which is calculated as the weighted linear combination of these filters as follows:

$$\Phi = \sum_{i=0}^{M-1}\left(E^{(i)}_{pass} + E^{(i)}_{stop}\right),$$

$$E^{(i)}_{pass} = \int W_p(\omega)(1 - |H_i(\omega)|)^2 d\omega$$

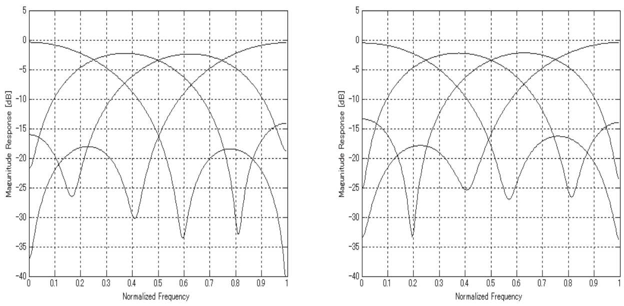
$$E^{(i)}_{stop} = \int W_s(\omega)|H_i(\omega)|^2 d\omega$$

where $i \in \{0,1]$ and $H(e^{j\omega})$ is an impulse response of the ith filter. $E^{(i)}_{pass}$ and $E^{(i)}_{stop}$ are called passband energy and stopband energy respectively. W_p and W_s are positive weighting function.



(a) (b)

Figure 6: Frequency responses of 4-channel order-2 prop. FB: (a) analysis bank (b) synthesis bank.



(a) (b)

Figure 7: Frequency responses of 4-channel order-2 conv. FB [9]: (a) analysis bank (b) synthesis bank.

In Fig. 6, we show the frequency responses of 4-channel order-2 proposed IIR PRFBs. These frequency responses have better stopband attenuation than the conventional FB in Fig.7. Also Table 1 compares stopband attenuations of the filterbanks. It is obvious that the proposed FBs have better performance than the conventional one in the same order. Also Fig. 8 shows frequency responses of the 8-channel order-2 proposed IIR PRFB.

5. CONCLUSION

In this paper, a new design approach of *M*-channel IIR PRFBs based on the IIR-FIR hybrid building blocks are presented. we impose the restriction in the building blocks to keep their order despite cascading of building blocks. Since our method has more free parameters than the conventional one, the proposed FBs have better stopband attenuation. As a feature, our IIR PRFBs have IIR analysis filters and FIR synthesis filters and the order of the synthesis filters are consistently.

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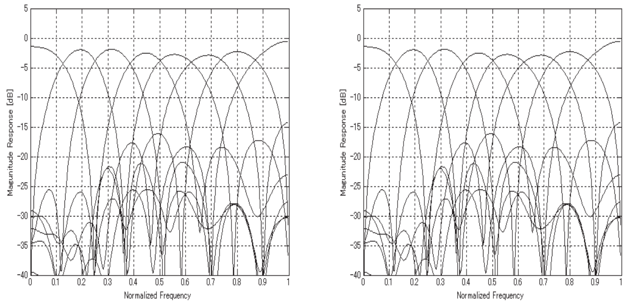
Table 1: Comparison of the stopband attenuations of 4-channel fiterbanks

	conv. FB in [9] (order-2)		prop. FB (order-1)		prop. FB (order-2)	
k	$H_k(z)$	$F_k(z)$	$H_k(z)$	$F_k(z)$	$H_k(z)$	$F_k(z)$
0	-18.5 dB	-16.3 dB	-17.1 dB	-17.2 dB	-22.5 dB	-25.6 dB
1	-14.0 dB	-13.9 dB	-12.8 dB	-12.6 dB	-23.5 dB	-24.6 dB
2	-16.0 dB	-13.4 dB	-13.7 dB	-13.8 dB	-27.0 dB	-27.0 dB
3	-18.1 dB	-17.9 dB	-18.3 dB	-18.2 dB	-23.8 dB	-20.6 dB



(a) (b)

Figure 8: Frequency responses of 8-channel order-2 prop. FB: (a) analysis bank (b) synthesis bank

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